2023-24 MATH2048: Honours Linear Algebra II

Homework 4

Due: 2023-10-09 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date. All questions are selected from Friedberg §2.4-2.5.

1. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible.

   Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible. (By this, the book means when A, B are not square, AB can still be invertible. No need to do this part.)

   Proof. Let L_A and L_B be two linear transformations from F^n to F^n corresponding to multiplication by A and B respectively. Given that AB is invertible, we can then say that $L_{AB} = L_A \circ L_B$ is an isomorphism.

   Since L_{AB} is injective, we can then infer that L_B is also injective, leading to the conclusion that $\ker(L_B) = 0$, which implies $\dim(\operatorname{im}(L_B)) = \dim(F^n) = n$.

   This leads to the conclusion that $\operatorname{im}(L_B) = F^n$, and hence L_B is bijective, and therefore an isomorphism.

   Consequently, $L_A = L_{AB}(L_B)^{-1}$ is an isomorphism. Therefore, both A and B are invertible. □

2. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

   Proof. Firstly, let's check that Φ is linear. If P and Q are two matrices, then

$$\Phi(cP + Q) = B^{-1}(cP + Q)B = cB^{-1}PB + B^{-1}QB = c\Phi(P) + \Phi(Q),$$

so Φ is indeed linear.

Secondly, define $\Psi(A) = BAB^{-1}$. It's easy to verify that $\Psi(\Phi(A)) = A$ and $\Phi(\Psi(A)) = A$ for all A in $M_{n\times n}(F)$, so Φ is invertible, and is an isomorphism with inverse Ψ. □

3. Let A be an $n \times n$ matrix.

(a) Suppose that $A^2 = 0$. Prove that A is not invertible.

(b) Suppose that $AB = 0$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

Proof. (a) Suppose that $A^2 = 0$. Assume that A is invertible. Then we have

$$A = A^{-1}A^2 = A^{-1} \cdot 0 = 0,$$

which is a contradiction. The zero matrix is not invertible. Therefore, A is not invertible.

(b) Suppose that $AB = 0$ for some nonzero $n \times n$ matrix B. Again, assume that A is invertible. Then we have

$$B = (A^{-1}A)B = A^{-1}(AB) = A^{-1} \cdot 0 = 0,$$

which is a contradiction since we assumed that B is a nonzero matrix. Therefore, A cannot be invertible. □

4. Let A and B be 2×2 matrices. Determine the matrix of the operator $T(M) = AMB$ on the space $F^{2\times 2}$ of 2×2 matrices, with respect to the basis $(e_{11}, e_{12}, e_{21}, e_{22})$ of $F^{2\times 2}$.

Proof. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be 2x2 matrices.

The basis of $F_{2\times 2}$ is given as $(e_{11}, e_{12}, e_{21}, e_{22})$, where e_{ij} is a 2x2 matrix with 1 at the (i, j) position and 0 elsewhere.

The effect of the operator $T(M) = AMB$ on the basis elements is as follows:

1. $Ae_{11}B = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ae & af \\ ce & cf \end{bmatrix}.$

2. $Ae_{12}B = A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ag & ah \\ cg & ch \end{bmatrix}$.

3. $Ae_{21}B = A \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ e & f \end{bmatrix} = \begin{bmatrix} be & bf \\ de & df \end{bmatrix}$.

4. $Ae_{22}B = A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ g & h \end{bmatrix} = \begin{bmatrix} bg & bh \\ dg & dh \end{bmatrix}$.

Therefore, the matrix representing the operator $T(M) = AMB$ on the space $F_{2\times2}$ with respect to the basis $(e_{11}, e_{12}, e_{21}, e_{22})$ is

$$\begin{bmatrix} ae & ag & be & bg \\ af & ah & bf & bh \\ ce & cg & de & dg \\ cf & ch & df & dh \end{bmatrix}.$$

□

5. Find all real 2×2 matrices that carry the line $y = x$ to the line $y = 3x$.

Proof. Let T be a linear transformation on R^2. Then T satisfies the given property if and only if $T(1, 1) = k(1, 3)$ for some non-zero k. This means that the matrices we are looking for are precisely those that satisfy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 3k \end{bmatrix}.$$

This equation yields a system of two linear equations:

$$a + b = k,$$
$$c + d = 3k.$$

Therefore, any matrix that sends the line $y = x$ to $y = 3x$ is of the form

$$\begin{bmatrix} a & k - a \\ c & 3k - c \end{bmatrix},$$

where a, c, and k are real numbers and k is non-zero. □